

Angular Momentum in Loop Quantum Gravity

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Abstract

An angular momentum operator in loop quantum gravity is defined using spherically symmetric states as a non-rotating reference system. It can be diagonalized simultaneously with the area operator and has the familiar spectrum. The operator indicates how the quantum geometry of non-rotating isolated horizons can be generalized to rotating ones and how the recent computations of black hole entropy can be extended to rotating black holes.

1 Introduction

In General Relativity familiar observables like energy or (angular) momentum, which are related to space-time symmetries, can be defined only in special regimes because in the general situation there is no reference frame with respect to which those symmetries could be defined. The usual procedure then is to introduce boundaries which are endowed with additional structure determined by suitable boundary conditions. Using the structure at the boundary, observables can be defined as functionals of the boundary values of the gravitational field.

As an example, recall the situation of angular momentum defined at spatial infinity of a space-time. Classically, one has to fix an asymptotic Cartesian frame carrying an $SO(3)$ -symmetry with respect to which angular momentum can be defined for an asymptotically flat space-time satisfying appropriate fall-off conditions. One can then read off the components of angular momentum by comparing the asymptotically flat metric with the fixed flat one or, in a Hamiltonian formulation, construct generators of rotations along the Killing vector fields of the flat metric [1]. In either way, one needs to fix a reference frame which is regarded as being non-rotating, and to specify fall-off conditions for asymptotically flat metrics, which build the subclass of space-times for which angular momentum at spatial infinity can be defined.

Asymptotic boundary conditions for gravity in Ashtekar's formulation have been discussed in Ref. [2]. In a connection formulation with its internal $SU(2)$ -gauge group, boundary conditions at spatial infinity not only involve specifying the fall-off of dynamical fields but also fixing an internal direction in $SU(2)$ -space because gauge transformations are frozen at spatial infinity. In polar coordinates (r, ϑ, φ) of the fixed reference frame of a

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non-compact space manifold Σ , the fall-off conditions for the connection state that all its components fall off at least with an order of r^{-2} at spatial infinity. Fixing the internal direction of the radial component A_r^i , which is the only one we need for the present purpose, is more crucial because, as we will see, it relates the internal spin to angular momentum. (At first sight it may seem alarming to relate angular momentum to operations in an internal gauge group, even more so because this is possible only if the internal group coincides with the group $SU(2)$ of rotations which could be a mere coincidence. Note, however, that in our case the internal gauge group of gravity is the local group of dreibein rotations which at the boundary are fixed and tied to global rotations by the phase space structure. In this context it is quite natural to have a relation between internal spins at the boundary and angular momentum.) The simplest choice may be to choose a constant internal direction (independent of ϑ, φ), but this is inappropriate for an asymptotically flat connection which, as discussed in Ref. [2], should have odd parity of its leading order term. We thus are lead to an asymptotic radial component of the connection having the form

$$A_r^i = r^{-2} A n_r^i(\vartheta, \varphi) + O(r^{-3}) \quad (1)$$

where A is independent of the polar coordinates and $n_r := (\sin \vartheta \cos \varphi, \sin \vartheta \sin \varphi, \cos \vartheta)$.

In light of our discussion above it is also worth mentioning that one can reverse the argumentation: starting from a boundary which topologically is a two-sphere but carries no additional structure, a suitable fixed internal direction (which should have mapping degree one) provides a bijective map $\partial\Sigma \rightarrow S^2$ which can be used to endow the boundary with an $SO(3)$ -action and thereby with a reference frame with respect to which angular momentum is to be defined. This view demonstrates why the fixed internal direction will play a prominent role in our definition of an angular momentum operator.

We next have to find a quantum formulation of the asymptotic structure by giving conditions for states to be considered as being asymptotically flat. Recalling that the kinematical Hilbert space of loop quantum gravity is a space of (cylindrical) functions on the space of generalized connections, the fall-off conditions can be imposed by constraining the support of an asymptotically flat state to only asymptotically flat connections. This immediately leads to the tangle property, which has been assumed in Ref. [3] in order to quantize the ADM energy, of asymptotically flat spin network states: an allowed spin network state has only transversal edges intersecting the boundary at infinity; for a family of holonomies h_r each lying in an orbit at radius r converges to unity for $r \rightarrow \infty$ because of the r^{-2} fall-off of the connection components, and a spin network state cannot depend non-trivially on this trivial holonomy.

It is, however, not immediate to see how the reference frame, with respect to which angular momentum will be defined, is realized in the spin network quantization. In order to illustrate its role we will first discuss possible standard approaches to a quantization of angular momentum. First, one can try to start from a classical expression of angular momentum, e.g. that of Ref. [2]

$$L_{\text{ADM}}[N^a] = \int_{\partial\Sigma} d^2y n_a N^{[b} E_i^{a]} K_b^i$$

associated with an asymptotic rotation generated by the vector field N^a , and then follow standard quantization steps. However, this expression contains the extrinsic curvature which would be quantized to a commutator of the volume operator with the Euclidean Hamiltonian constraint [4] resulting in a very complicated operator.

A second procedure could be to start from the simple action of rotations on spin networks (simply rotating the graph; this is the usual action of the diffeomorphism group where, however, rotations are not included in the gauge group of diffeomorphisms because those have to be asymptotically trivial) and to determine angular momentum as their generators by differentiation. However, this does not work because as with the diffeomorphism constraint the action used on the space Φ_Σ of cylindrical functions is not strongly continuous and so its generators do not exist [5, Appendix C].

2 The Angular Momentum Operator

Our proposal to remedy this situation is to use spherically symmetric states [6] which are distributional states in the kinematical Hilbert space of loop quantum gravity being exactly symmetric with respect to a given action of the rotation group (classified by a conjugacy class $[\lambda]$ of homomorphisms from the $U(1)$ -isotropy group of a general point in Σ to the gauge group $SU(2)$). All those symmetric states, which are defined as distributions being supported on the space of rotationally invariant connections, can be identified with cylindrical functions of connections and scalar fields on a radial manifold B . Roughly, this comes from the decomposition (up to gauge transformations)

$$\begin{aligned} A(r, \vartheta, \varphi) &= A_1(r) n_r^i \tau_i dr + 2^{-\frac{1}{2}} (A_2(r) n_\vartheta^i + (A_3(r) - \sqrt{2}) n_\varphi^i) \tau_i d\vartheta \\ &\quad + 2^{-\frac{1}{2}} (A_2(r) n_\varphi^i - (A_3(r) - \sqrt{2}) n_\vartheta^i) \tau_i \sin \vartheta d\varphi \end{aligned}$$

of an invariant connection into a reduced connection given by A_1 and scalar fields A_2, A_3 ; for details we refer to Ref. [6]. Vice versa, any cylindrical state in the full theory can be mapped (“averaged”) to a distributional state in the reduced formulation giving rise to a map $\rho_{[\lambda]}: \Phi_\Sigma \rightarrow \Phi'_B$ with Φ'_B denoting the topological dual to the space of cylindrical functions on the space of generalized connections and scalar fields in B . This map is defined by viewing a given cylindrical function $f \in \Phi_\Sigma$ as a function defined only on the subspace of invariant connections, and can be considered as a pull back of cylindrical functions to the space Φ'_B . Employing $\rho_{[\lambda]}$ and the pull back to Φ'_B of general states serves two purposes: On the one hand, evaluated in an invariant connection all holonomies to rotated asymptotic edges, which are transversal to the orbits owing to the tangle property, are gauge equivalent because connections in the support of the pull backs are invariant under rotations up to gauge. This implies that rotated spin networks are gauge equivalent rather than orthonormal, and so infinitesimal generators of rotations exist being related to generators of internal rotations by the fixed asymptotic internal directions. On the other hand, by using the map $\rho_{[\lambda]}$, which is a central ingredient in specifying symmetric states with respect to an action of $SO(3)$, we introduce a reference frame with respect to which angular momentum will be defined.

Denoting the action of a rotation by an angle δ around some given axis v as $R(v, \delta)$, the derivative $\frac{d}{d\delta} R(v, \delta) T_I$ of the action on a spin network state T_I does not exist. Instead, we are going to define an angular momentum operator \hat{L}_v by

$$\rho_{[\lambda]}(\hat{L}_v T_I) := -i\hbar \left(\frac{d}{d\delta} \rho_{[\lambda]}(R(v, \delta) T_I) \right) \Big|_{\delta=0} \quad (2)$$

using a different ordering of the pull back $\rho_{[\lambda]}$ and the derivative in order to render the derivative existing.

To derive the action of L_v explicitly, we write a spin network state as $T_I^{a_1, \dots, a_n}$ which has n punctures at the sphere at infinity, each carrying an index a_p in the representation with label j_p of the intersecting edge (all edges are assumed to be outgoing at infinity). Note that internal gauge transformations are frozen at the asymptotic boundary, and therefore each state is a vector valued function on the space of generalized connections taking values in the tensor product $j_1 \otimes \dots \otimes j_n$. For the pull back to invariant connections only the radial part of an asymptotic holonomy, which is transversal due to the tangle property, matters and has the form

$$h_e = \exp \left(- \int_e dr A(r) n_r^i(\vartheta, \varphi) \tau_i \right).$$

After a rotation by an angle δ around the polar axis v_3 of the coordinates it changes by conjugation to

$$h'_e = \exp \left(- \int_e dr A(r) n_r^i(\vartheta, \varphi + \delta) \tau_i \right) = \exp(\delta \tau_3) h_e \exp(-\delta \tau_3).$$

The exponential at the left hand side of h_e corresponds to an $SU(2)$ -transformation at the puncture at infinity, whereas the exponential at the right hand side corresponds to an inner vertex and is absorbed due to gauge invariance. We, therefore, have

$$\rho_{[\lambda]}(R(v_3, \delta) T_I^{a_1, \dots, a_n}) = \pi^{j_1}(\exp(\delta \tau_3))_{b_1}^{a_1} \dots \pi^{j_n}(\exp(\delta \tau_3))_{b_n}^{a_n} \rho_{[\lambda]}(T_I^{b_1, \dots, b_n})$$

because the p -th index is in the representation π^{j_p} .

This expression can easily be differentiated with respect to δ yielding by inspection of the general definition (2)

$$\hat{L}_3 T_I = -i\hbar (\pi^{j_1}(\tau_3) \oplus \dots \oplus \pi^{j_n}(\tau_3)) T_I \quad (3)$$

as the third component of the angular momentum operator generating rotations around the z -axis (which has been used as axis for the polar coordinates). A rotation around an arbitrary axis given by the direction v^i in S^2 leads to a conjugation of asymptotic holonomies with $\exp(\delta v^i \tau_i)$ and the angular momentum with respect to this direction is

$$\hat{L}_v T_I = -i\hbar (\pi^{j_1}(v^i \tau_i) \oplus \dots \oplus \pi^{j_n}(v^i \tau_i)) T_I. \quad (4)$$

Taking the three components $\hat{L}_1, \hat{L}_2, \hat{L}_3$ corresponding to rotations around Cartesian axes, this immediately implies the correct commutation relations (which, of course, directly come from the relation of rotations to internal rotations) of an angular momentum:

$$[\hat{L}_i, \hat{L}_j] = -i\hbar \epsilon_{ijk} \hat{L}_k. \quad (5)$$

Furthermore, we can determine the angular momentum spectrum: we just have to decompose the tensor product of all representations associated with punctures at infinity into irreducible ones by building appropriate linear combinations of the components $T_I^{a_1, \dots, a_n}$ which transform under one irreducible subrepresentation. Any component of angular momentum then has eigenvalues $\hbar m$ where $m = \sum_i m_i \in \frac{1}{2}\mathbb{Z}$ is given by a sum over all punctures each of which transforms like a state in the j_i -representation given by m_i . Also the absolute value has the usual eigenvalues $L^{(j)} = \hbar \sqrt{j(j+1)}$, $j \in \frac{1}{2}\mathbb{N}_0$ with eigenstates

being given by spin network states. In particular, the absolute value of the angular momentum operator and the area operator are simultaneously diagonalizable. Furthermore, for a given spin network having a set of spins $\{j_p\}$ labeling its punctures at infinity, an upper bound for the angular momentum eigenvalues is given by

$$L \leq \hbar \sqrt{\sum_p j_p (\sum_p j_p + 1)}. \quad (6)$$

3 Inequalities between Angular Momentum and Area

As argued in Ref. [7], an inequality like the last one has an immediate application to extremal black holes which are classically defined as saturating the “no naked singularity” condition $L \leq (8\pi G)^{-1}A$ between angular momentum and horizon area of a Kerr black hole. Because the author of Ref. [7] had no angular momentum operator at his disposal, he assumed (without any concrete justification) the angular momentum eigenvalues to be given by the spins of a spin network satisfying the inequality $L \leq \hbar \sum_p j_p$ (although not noted explicitly, it has also been assumed that angular momentum and area are simultaneously diagonalizable). Using that, for a given set of punctures, the area eigenvalues $A = 8\pi\gamma l_P^2 \sum_p \sqrt{j_p(j_p + 1)}$ are bounded from below by $8\pi\gamma l_P^2 \sum_p j_p$, an inequality $L \leq (8\pi\gamma G)^{-1}A$ for the eigenvalues was derived which resembles the classical relation. Because the area eigenvalue for a given $\sum_p j_p$ is minimal if there is only one puncture with spin $\sum_p j_p$, this one-puncture case was identified with an extremal black hole.

We will now see what the situation looks like when using our angular momentum operator. However, there is the important caveat that the operator (3) has been defined at infinity and not, as needed in Krasnov’s argumentation, at the horizon. Noting that all we needed for our definition of the angular momentum operator was the asymptotic form of a connection and the fact that an internal direction of mapping degree one is fixed at the boundary, we can immediately apply our discussion of spatial infinity to the case of an isolated horizon: the isolated horizon boundary conditions also can be used to fix an internal direction of mapping degree one at the horizon two-sphere (see, e.g., Refs. [8, 9]). Although internal gauge transformations are no longer frozen at an inner boundary, the boundary punctures of a bulk spin network are the same as used above because they are coupled to a Chern–Simons state at the boundary which restores gauge invariance. All the remaining steps of the derivation of the angular momentum operator then go through unaltered.

Now we can compare the eigenvalues of angular momentum and horizon area. First, we have a larger upper bound for angular momentum than assumed in Ref. [7], given by Eq. (6). However, the lower bound for the area eigenvalues associated with a set of punctures $\{j_p\}$ can be refined by using the inequality

$$\begin{aligned} \sqrt{(j_1 + j_2)(j_1 + j_2 + 1)} &= \sqrt{j_1^2 + j_2^2 + 2j_1j_2 + j_1 + j_2} \\ &\leq \sqrt{j_1(j_1 + 1) + 2\sqrt{j_1(j_1 + 1)j_2(j_2 + 1)} + j_2(j_2 + 1)} \\ &= \sqrt{j_1(j_1 + 1)} + \sqrt{j_2(j_2 + 1)} \end{aligned}$$

and induction over the number of punctures. This again leads to an inequality

$$\hbar^{-1}L \leq \sqrt{\sum_p j_p (\sum_p j_p + 1)} \leq (8\pi\gamma l_P^2)^{-1}A \quad (7)$$

which is saturated for one-puncture states and differs from the classical relation only by the factor γ . Note that only the area spectrum is affected by this parameter, whereas the spectrum of angular momentum is protected against a rescaling by the commutation relations.

4 Non-Rotating and Rotating Isolated Horizons

As another application of the angular momentum operator, we check whether the non-rotating horizon geometry derived in Ref. [8] corresponds to zero angular momentum as seen from the quantum theory. Recall that non-rotating isolated horizons are defined classically by suitable boundary conditions which then are used to select a sector of space-times to be quantized. Classically the condition of being non-rotating is implemented by requiring the intrinsic geometry of the horizon to be spherically symmetric. After quantization, the quantum horizon geometry is described by a “punctured sphere” where a spin network in the bulk (outside the horizon) pierces the horizon in isolated punctures thereby providing the horizon two-sphere with geometry. Note, incidentally, that the quantum geometry of a non-rotating horizon is no longer spherically symmetric which coincides with the observation [6] that exactly spherically symmetric states only exist in the sense of distributions: the quantum horizon geometry of a realistic black hole cannot be spherically symmetric even if it is non-rotating. Even finest approximations of a symmetric distribution by ordinary states will break the symmetry by introducing a discrete set of punctures.

In the framework of Ref. [8] the boundary degrees of freedom of a non-rotating isolated horizon are described by a Chern–Simons theory which is “glued” to the bulk spin network by a boundary condition. A bulk state is labeled by the spins $j = (j_1, \dots, j_n) \in (\frac{1}{2}\mathbb{N})^n$ of the punctures together with half-integers $m = (m_1, \dots, m_n) \in (\frac{1}{2}\mathbb{Z})^n$ which determine a state in the product of all representations labeling the punctures, i.e., they are subject to the conditions $m_i \in \{-j_i, -j_i + 1, \dots, j_i\}$. Given such a combination of spin labels of the bulk state, a permissible boundary state of the Chern–Simons theory is determined by numbers $a = (a_1, \dots, a_n) \in \mathbb{Z}_k^n$ fulfilling $a_i \equiv -2m_i \pmod k$ and $\sum_i a_i \equiv 0 \pmod k$ where $k = \frac{a_0}{4\pi\gamma l_P^2}$ is the level of the Chern–Simons theory and related to the classical horizon area a_0 (a prescribed parameter). The first condition on a describes how the bulk states labeled by (j, m) are glued to a Chern–Simons state labeled by a , and the second condition arises from gauge invariance in Chern–Simons theory and can be interpreted as saying that the sum of all deficit angles introduced by the punctures vanishes modulo 4π [9].

Using our angular momentum operator, we can see another interpretation of the condition $\sum_i a_i \equiv 0$ if we first transfer it via the gluing conditions to the bulk labels resulting in $\sum_i m_i = 0$. Here we ignored the fact that the sum of the a_i vanishes only modulo k which can be seen only in the Chern–Simons boundary theory. However, for any combination of labels m_i subject to the condition $\sum_i m_i = 0$ we can find a permissible set of labels a_i . The condition $\sum_i m_i = 0$ in turn says that the bulk state can be associated with the trivial representation in the tensor product of all puncture representations and so has vanishing angular momentum as measured with the operator (3). Thus, the condition $\sum_i a_i \equiv 0$ on the non-rotating quantum horizon state can be interpreted naturally as saying that the classical property of being non-rotating is preserved after quantization.

Our interpretation of the boundary states also indicates how the quantum geometry of non-rotating horizons could be generalized to rotating ones. In fact, on a bulk state we just

have to replace the condition $\sum_i m_i = 0$ by a condition $\sum_i m_i = l_0 \neq 0$ where l_0 is a given (analogously to the classical horizon area a_0) value of the angular momentum. However, this argument tells nothing about how to generalize the Chern–Simons boundary theory to rotating isolated horizons which could only be derived by a Hamiltonian analysis using more general boundary conditions.

Assuming that the generalization to rotating horizons is correct we can check whether the successful calculation of the entropy of non-rotating black holes (possibly charged and non-extremal) [9] remains valid for rotating black holes. In fact, it is quite easy to see that this is the case for rotating black holes not too close to extremality $((8\pi\gamma l_P^2)^{-1}a_0 \gg l_0)$ using the methods of Ref. [9]: First, a lower bound for the number of states with prescribed area around a_0 and angular momentum around $\hbar l_0$ can be derived by using configurations j with $j_1 = \dots = j_{n-1} = \frac{1}{2}$ and $j_n = l_0$ together with $m_1 = \dots = m_{n-1} = \pm \frac{1}{2}$ and $m_n = l_0$. The condition $\sum_{i=1}^n m_i = l_0$ then is equivalent to $\sum_{i=1}^{n-1} m_i = 0$ and results in a number

$$N_{\text{bh}} \geq \frac{2^{n-\frac{1}{2}}}{\sqrt{\pi(n-1)}}$$

of states (just replace n with $n-1$ in the corresponding formula of Ref. [9]). Taking the logarithm and using the condition of not being near-extremal we obtain the same lower bound for the entropy

$$S_{\text{bh}} \geq \frac{\log 2}{4\pi\sqrt{3}\gamma l_P^2} a_0 - o(a_0)$$

as in Ref. [9]. Because the derivation of an upper bound for the entropy in Ref. [9] did not make use of the condition $\sum_i a_i \equiv 0$, we can immediately transfer it to the rotating case and arrive at our result that the recent calculations of black hole entropy in loop quantum gravity remain valid without changes also for rotating (possibly charged but far from extremal) black holes:

$$S_{\text{bh}} = \frac{\log 2}{4\pi\sqrt{3}\gamma l_P^2} a_0 + o(a_0). \quad (8)$$

In particular, the Bekenstein–Hawking formula can be obtained with the correct numerical factor not only for charged but also for rotating black holes by fixing the Immirzi parameter to be

$$\gamma_0 = \frac{\log 2}{\pi\sqrt{3}}. \quad (9)$$

For near-extremal rotating black holes, however, the entropy is reduced. In the extremal case we have $a_0 = 8\pi\gamma l_P^2 \sqrt{l_0(l_0+1)}$ and according to Ref. [7] the boundary state has a single l_0 -puncture with at most $2l_0+1$ values for m . This results in $S_{\text{extr}} \leq \log((4\pi\gamma l_P^2)^{-1}a_0)$ being at most logarithmic in the area.

To conclude, we note that as demonstrated here the methods developed in Ref. [6] are not only applicable in the study of reduced models but also provide tools for direct applications to the full theory.

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